RESEARCH STATEMENT

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My research is primarily concerned with probability theory and its applications. I am particularly interested in the convergence behavior of Markov chains defined on discrete combinatorial and algebraic structures. I also enjoy working on distributional approximation using Stein's method techniques.

Markov Chains.

A major theme in my research is the determination of non-asymptotic convergence rates of random processes. For example, suppose that $\{X_k\}_{k=0}^{\infty}$ is an ergodic Markov chain with finite state space $\mathcal{X}$, transition probabilities $P(x,y) = \mathbb{P}\{X_{k+1} = y \,|\, X_k = x\}$, and stationary distribution π. It is classical that the exponential rate of convergence is governed by the subdominant eigenvalue of the transition operator, but this information is generally not sufficient for describing the short term mixing behavior. In practice, one would like to have sharp upper and lower bounds on $d\left(P_\mu^k, \pi\right)$, where d is a suitable metric on the space of probability measures on $\mathcal{X}$ and P_μ^k is the distribution of X_k given that $X_0 \sim \mu$. In particular, given $\varepsilon > 0$, it is of interest to estimate $t_{mix}^*(\varepsilon) = \inf\left\{k \in \mathbb{N} : \sup_\mu d\left(P_\mu^k, \pi\right) < \varepsilon\right\}$, the number of steps required for the chain to be within ε of stationarity for any initial distribution. In many cases there is a sharp transition to equilibrium in the sense that $\sup_\mu d\left(P_\mu^k, \pi\right)$ stays close to its maximum value for some time and then abruptly drops and tends rapidly to zero. Though this "cut-off phenomenon" appears to be quite common in most cases of interest, the nature of its existence remains one of the central mysteries in the modern theory of Markov chains.

The standard example of this line of inquiry is the question "How many shuffles are required to mix up a deck of n cards?" This is generally interpreted in terms of a Markov chain on the symmetric group where $\sigma \in S_n$ represents the arrangement with $\sigma(j)$ the label of the card in position j. Often, the chain evolves as a random walk on S_n, so that after k shuffles the state of the deck initially ordered as $X_0 = \sigma_0$ is $X_k = \sigma_0 \sigma_1 \cdots \sigma_k$ where the $\sigma_i's$ are chosen independently from some probability measure on S_n. Under mild assumptions on the measure driving the walk (which determines the type of shuffle being performed), the stationary distribution will be uniform, corresponding to a completely "mixed up" deck of cards. Here, cut-off is equivalent to the existence of a sequence $t_n \to \infty$ such that, as n goes to infinity, the distance to uniformity after $(1+\varepsilon)t_n$ shuffles of an n-card deck approaches zero for $\varepsilon > 0$ and approaches its maximum for $\varepsilon < 0$. In this case, it can be said that t_n shuffles are necessary and sufficient to mix a deck of n cards. A problem of this sort that I am particularly interested in right now involves the random-to-random insertion shuffle, which proceeds by choosing a card at random and then moving it to a random location. Though the dynamics are quite simple, the question of whether this chain exhibits total variation cut-off has been open for several decades and it seems that its resolution will require some novel ideas.

The broad goal of these investigations is to gain a deeper understanding of how random processes equilibrate. Beyond the intrinsic interest of this central question, one of the main attractions for me is that these seemingly straightforward problems are often quite challenging and involve the interplay of diverse areas of mathematics, ranging from classical probability and linear algebra to representation theory and PDEs. Moreover, the techniques and results have found a wide variety of applications throughout the sciences, so there are plenty of opportunities for interdisciplinary collaboration. Finally, in order to obtain sharp results one generally has to really get to know the chain in question and understand what lies at the heart of its mixing behavior. For this reason, I am often drawn to examples in which the state space is interesting in its own right. For excellent overviews of various aspects of Markov chain mixing, I highly recommend [23, 15, 37, 29].

Abelian Sandpiles

A recent example of my work with Markov chains involves random walks on abelian sandpiles. These objects were introduced by physicists in the late eighties as a toy model for self-organized criticality [4], and have since appeared in a wide variety of contexts [24]. The general set-up is a simple connected graph G with vertex set $V = \{v_1, ..., v_{n-1}, v_n = s\}$. The vertex s is called the sink. A sandpile is a collection of indistinguishable chips distributed amongst the non-sink vertices $\widetilde{V} = V \setminus \{s\}$, and thus can be represented as a function $\eta : \widetilde{V} \to \mathbb{N}_0$. The configuration η is stable if $\eta(v) < \deg(v)$ for each $v \in \widetilde{V}$. If $\eta(v) \geq \deg(v)$, then v is allowed to topple, sending a chip to each of its neighbors. This may cause other sites to become unstable, leading to further topplings, and chips falling into the sink are lost forever. The nomenclature derives from the fact that the final stable configuration does not depend on the order in which topplings take place. (The presence of the sink ensures that any configuration can be stabilized by performing finitely many topplings at unstable sites.) A natural Markov chain on stable sandpiles evolves at each time step by adding a chip to a random vertex and stabilizing. Since the set $\mathcal{S}$ of stable configurations equipped with the relation of pointwise addition followed by stabilization has the structure of a commutative monoid [3], the process can be interpreted as a random walk on $\mathcal{S}$ driven by the uniform distribution on single-chip configurations. The recurrent communicating class of this chain is the minimal ideal $\Gamma = \eta_* + \mathcal{S}$ where $\eta_*(v) = \deg(v) - 1$. As the kernel of a commutative semigroup, Γ forms an abelian group, and the sandpile chain restricted to recurrent states can be regarded as a random walk on Γ.

In joint work with Dan Jerison and Lionel Levine [21], we have found a way to compute the eigenvalues and eigenfunctions of the transition operator in terms of functions $h : V \to \mathbb{T}$ which satisfy a mean value property with respect to the geometric mean along with a boundary condition at the sink. Specifically, define the set of *multiplicative harmonic functions* on G by

$$\mathcal{H} = \big\{ h : V \to \mathbb{T} \text{ such that } h(v)^{\deg(v)} = \prod_{u \sim v} h(u) \text{ and } h(s) = 1 \big\}.$$

Then $\mathcal{H}$ is finite with cardinality equal to the number of spanning trees in G, and we have

Theorem 1. *For $h \in \mathcal{H}$, define $f_h : \Gamma \to \mathbb{T}$ by $f_h(\eta) = \prod_{i=1}^n h(v_i)^{\eta(v_i)}$. Then $\{f_h\}_{h \in \mathcal{H}}$ is an orthonormal basis of eigenfunctions for the sandpile chain on G. The eigenvalue corresponding to f_h is $\lambda_h = \frac{1}{n} \sum_{i=1}^n h(v_i)$.*

The choice of generators for the random walk ensures that the sandpile chain is irreducible and aperiodic, so it follows from classical theory that the k-step distributions converge to the uniform measure on recurrent configurations as $k \to \infty$. (In fact, our formula for the eigenvalues generalizes to arbitrary chip addition distributions and shows that the chain is aperiodic even if we do not allow chips to be added to the sink.) Using Fourier methods and other techniques from Markov chain theory, we are able to bound the mixing time in terms of this spectral information. For example, we showed that the sandpile chain on the complete graph K_n exhibits total variation cut-off at time $\frac{1}{4\pi^2} n^3 \log(n)$, whereas the sandpile chain mixes instantly on the cycle C_n, and in between order n and $n \log(n)$ steps on the torus $\mathbb{Z}_m \times \mathbb{Z}_m$ with $n = m^2$ vertices.

For general graphs, computing the mixing time can be quite delicate. However, up to a log factor of the order of the sandpile group, the mixing time is determined by the maximum modulus of the nontrivial eigenvalues of the transition matrix, and we show that

Theorem 2. *If $G = (V, E)$ is a simple connected graph with $|V| = n$ and $\max_{v \in \widetilde{V}} \deg(v) = d$, then every nontrivial eigenvalue of the sandpile chain on G satisfies $|\lambda| \leq 1 - \dfrac{8}{d^2 n}$.*

Theorem 2 implies that the mixing time of the sandpile chain is $O\left(d^2 \log(d) n^2\right)$. A more involved argument which takes all eigenvalues into account and uses facts about the *smoothing parameter* from [27] gives an improved bound of $O\left(d^2 n \log(n)\right)$.

An interesting fact that has emerged from our investigations is that one can sometimes lower bound the subdominant eigenvalue (and thus the mixing time) by checking for certain substructures in the underlying graph. The idea is that large eigenvalues correspond to nearly constant multiplicative harmonic functions. We say that a graph $H = (W, F)$ is a *gadget* of size m if there is a function $\widetilde{h}$ on W which maps a proper subset $W^\circ \subset W$ with $|W^\circ| = m$ to $\mathbb{T} \setminus \{1\}$, maps $W \setminus W^\circ$ to 1, and satisfies $\widetilde{h}(v)^{\deg(v)} = \prod_{u \sim v} \widetilde{h}(u)$ for all $v \in W$. If (W, F) is an induced subgraph of G with boundary contained in $W \setminus W^\circ$, then $\widetilde{h}$ extends to a multiplicative harmonic function on G which maps $V \setminus W^\circ$ to 1. When $m \in O(1)$, this implies that the spectral gap of the sandpile chain is $O\left(n^{-1}\right)$, which matches the lower bound from Theorem 2 for bounded degree graphs.

A typical example of a gadget is the following: If $v_1, v_2 \in V$ have the same set of neighbors, N, then $\{v_1, v_2\} \cup N$ gives a gadget of size 2. There are $|N| - 1$ nontrivial multiplicative harmonic functions which map $V \setminus \{v_1, v_2\}$ to 1, defined by setting $h_\omega(v_1) = \omega$, $h_\omega(v_2) = \omega^{-1}$ for ω a nontrivial $|N|th$ root of unity. One reason gadgets are intriguing is that they show that, unlike ordinary random walks, one can sometimes dramatically slow down the mixing time by "wiring in" a small gadget off in some corner. Also, being able to infer global properties from local information can be extremely useful for complicated graphs. For instance, we were able to use the gadget paradigm to compute the order of the spectral gap of the sandpile chain on Sierpinski gasket graphs despite the fact that no one has yet been able to deduce the structure of the corresponding sandpile group.

A final feature of our theory which deserves mention is that one can bound the sizes of eigenvalues of sandpile chains in terms of the lengths of certain vectors in appropriate "dual lattices." As a consequence of this correspondence, we have discovered an inverse relationship between the spectral gap of the sandpile chain and that of ordinary random walk on the underlying graph:

Theorem 3. *Let β denote the random walk spectral gap of a graph G on n vertices. Then the spectral gap of the sandpile chain on G satisfies $\gamma \leq \dfrac{4\pi^2}{\beta n}$.*

This shows, for example, that the sandpile chain on a bounded degree expander graph has the smallest possible gap of all graphs with the same number of vertices and maximal degree. It also enables us to estimate the mixing time of sandpile chains on certain families of random graphs such as Erdős-Rényi graphs and random d-regular graphs.

Hyperplane Arrangements

I am also interested in extending my thesis work on random walks on the chambers of hyperplane arrangements. The basic idea here is that one has a finite collection of codimension-1 subspaces of $\mathbb{R}^n$ which carve the ambient space into disjoint pieces called faces. The n-dimensional faces are called chambers and there is a natural semigroup product on the set of faces with respect to which the chambers form a two-sided ideal. By endowing the faces with a probability measure, one can construct a random walk on the chambers in terms of repeated left-multiplication by randomly chosen faces. Much of the interest in hyperplane chamber walks lies in the fact that a surprisingly diverse assortment of Markov chains can be represented within this framework. Examples include models of conquering territories, coupon collecting, and queuing systems, as well as a large number of card shuffling schemes and other natural random walks on finite Coxeter groups. This construction was first introduced by Pat Bidigare in 1997 [9], and subsequent investigations have yielded a rich and fascinating theory, showing that the transition matrices are diagonalizable over $\mathbb{R}$ with explicitly computable eigenvalues

and establishing a criterion for ergodicity along with a description of the stationary distributions and upper bounds on convergence rates [8, 11]. A general description of the left eigenspaces of an associated operator in the face semigroup algebra has also been given [14], and many of these ideas have been further generalized to the setting of oriented matroids [11], left-regular bands [10, 36], and $\mathscr{R}$-trivial monoids [2].

In my dissertation [32], I introduced an alternative description of these hyperplane walks in terms of stacking rows of colored tiles which allows for easy visualization and admits various generalizations. Using this perspective, I was able to give an explicit construction of the right eigenfunctions corresponding to the largest eigenvalues (along with a general prescription for finding others) by projecting the walk onto various subarrangements. The importance of eigenfunctions lies in the fact that the k-step distributions can be characterized by the expectations they assign to various functions. For a diagonalizable chain, such functions can be expanded in an eigenbasis and the eigenvalues tell us exactly how the different components decay as the chain evolves. Since so many interesting Markov chains can be represented as random walks on hyperplane arrangements, a general theory of their eigenfunctions provides valuable information in a wide range of problems.

To connect with the introductory example, many common classes of shuffling procedures (such as biased a-shuffles, riffle shuffles with fixed cut size, and top-m-to-random shuffles) can be modeled in terms of random walks on the braid arrangement driven by measures which are invariant under transpositions, and it follows from my work on eigenfunctions for hyperplane walks that

Theorem 4. *If w is a probability measure on the faces of the braid arrangement which is invariant under transpositions, then*

$$\phi_{(i,j)}(\pi) = 1\{\pi^{-1}(i) < \pi^{-1}(j)\} - \frac{1}{2},\ 1 \leq i < j \leq n$$

are linearly independent eigenfunctions corresponding to the subdominant eigenvalue of the associated random walk on S_n.

As an application of the preceding theorem, consider the shuffle which proceeds by cutting off the top m cards and inserting them randomly amongst the remaining $n - m$ cards while retaining their relative order. Let P^k denote the distribution of the deck after k shuffles of this type (starting with an ordered deck, say) and U the uniform distribution over all possible arrangements. Using Wilson's method [40] and Theorem 4, I showed that

Theorem 5. *If* $k \geq \dfrac{n(n-1)}{4m(n-m)} \log(n) - c\dfrac{n(n-1)}{m(n-m)}$, *then* $\left\| P^k - \pi \right\|_{TV} \geq 1 - 3e^{3m-4c}$.

For m fixed, this matches the best known upper bound on mixing time up to a factor of 4 [16].

Markov chain eigenfunctions can also be used to bound mixing times from above. For example, I showed that hyperplane chamber walks are stochastically monotone with respect to various generalizations of the weak Bruhat order and used eigenfunctions in conjunction with coupling arguments from [22] to obtain upper bounds in certain cases. Specifically, if $\{H_i\}_{i=1}^m$ is a central arrangement of hyperplanes in $\mathbb{R}^n$, writing $\sigma_i(F)$ for the ith sign sequence coordinate of a face F (which indicates its location relative to H_i), and λ_i for the eigenvalue indexed by the flat H_i, one has

Theorem 6. *Let P be the transition operator for an ergodic hyperplane chamber walk with stationary distribution π. If the hyperplane eigenvalues satisfy $\lambda_1 = ... = \lambda_m = \lambda \in (0, 1)$, then for any $k \in \mathbb{N}$ and any initial state $C_0 \in \mathcal{C}$,*

$$\left\| P_{C_0}^k - \pi \right\|_{TV} \leq \lambda^k \sum_{i=1}^{m} \mathbb{P}\{\sigma_i(Y) \neq \sigma_i(C_0)\}, \quad Y \sim \pi.$$

Stein's Method.

My other main research interest at this point involves studying the convergence behavior of sequences of random variables using an approach known as Stein's method. Stein's method refers to a general framework based on solutions of certain differential or difference equations for bounding the distance between the distribution of a random variable X and that of a random variable Z having some specified target distribution. The metrics for which this approach is applicable are of the form $d_{\mathcal{H}}\left(\mathscr{L}(X), \mathscr{L}(Z)\right) = \sup_{h \in \mathcal{H}} |\mathbb{E}\left[h(X)\right] - \mathbb{E}\left[h(Z)\right]|$ for some suitable class of functions $\mathcal{H}$, and include the Kolmogorov, Wasserstein, and total variation distances as special cases. The basic idea is to find an operator $\mathcal{A}$ such that $\mathbb{E}\left[(\mathcal{A}f)(X)\right] = 0$ for all f belonging to some sufficiently large class of functions $\mathcal{F}$ if and only if $\mathscr{L}(X) = \mathscr{L}(Z)$. (For example, Charles Stein proved that $Z \sim \mathcal{N}(0,1)$ if and only if $\mathbb{E}\left[Zf(Z) - f'(Z)\right] = 0$ for all Lipschitz f [39].) If one can then show that for any $h \in \mathcal{H}$, the equation $(\mathcal{A}f)(x) = h(x) - \mathbb{E}\left[h(Z)\right]$ has solution $f_h \in \mathcal{F}$, then upon taking expectations, absolute values, and suprema, one finds that

$$d_{\mathcal{H}}\left(\mathscr{L}(X), \mathscr{L}(Z)\right) = \sup_{h \in \mathcal{H}} |\mathbb{E}\left[h(X)\right] - \mathbb{E}\left[h(Z)\right]| = \sup_{h \in \mathcal{H}} |\mathbb{E}\left[(\mathcal{A}f_h)(X)\right]|.$$

Remarkably, it is often easier to work with the right-hand side of this equation and the techniques for analyzing distances between probability distributions in this manner are collectively known as Stein's method. Wonderful surveys of the subject can be found in [35, 13].

One of the earliest Stein's method techniques for estimating the error in normal approximation involves the construction of an exchangeable pair (W, W') such that $\mathbb{E}\left[W' | W\right] = (1 - \lambda)W$ for some $\lambda \in (0, 1)$. A common method of obtaining these *Stein pairs* is to take successive steps of a reversible Markov chain in equilibrium. In [19], Jason Fulman used a non-reversible Markov chain to construct a Stein pair with which to bound the rate of normal convergence for the number of descents and inversions in a random permutation. My initial foray into the subject was concerned with generalizing Fulman's work to treat the number of d-descents in a random permutation. (For any $1 \le d \le n - 1$, a pair (i, j) is said to be a d-descent of $\sigma \in S_n$ if $i < j$ and $\sigma(j) < \sigma(i) \le \sigma(j) + d$, and the number of d-descents is denoted $\mathrm{Des}_{n,d}(\sigma)$. The cases $d = 1$ and $d = n - 1$ correspond to descents and inversions, respectively.) By using a similar construction as [19] and appealing to a result due to Adrian Röllin regarding the necessity of exchangeability assumptions in Stein bounds [34], I proved

Theorem 7. *For any fixed $d \in \mathbb{N}$, there is a universal constant $M = M(d)$ such that for all $n \ge d$, $x \in \mathbb{R}$,*

$$\left| \mathbb{P}\left(\frac{\mathrm{Des}_{n,d} - \mu_{n,d}}{\sigma_{n,d}} \le x\right) - \Phi(x) \right| \le \frac{M}{\sqrt{n}}$$

where Φ is the standard normal c.d.f., $\mu_{n,d} = \mathbb{E}\left[\mathrm{Des}_{n,d}\right]$, and $\sigma_{n,d} = \sqrt{\mathrm{Var}(\mathrm{Des}_{n,d})}$.

I established similar bounds when d grows as a function of n. Interestingly, my later work on eigenfunctions for hyperplane walks showed that for all $1 \le d < n$, $\mathrm{Des}_{n,d} : S_n \to \mathbb{N}_0$ is an eigenfunction for the subdominant eigenvalue of a large class of random walks on S_n that includes the random-to-end shuffle, a fact which lies at the heart of the arguments in [19, 31].

Another example of my work on Stein's method is the paper [33], joint with Haining Ren, in which we established the first Stein's method framework for Laplace approximation and applied it to the study of random sums. We began by showing that a random variable X has the symmetric Laplace distribution with mean zero and variance $2b^2$ if and only if $\mathbb{E}\left[g(X)\right] - g(0) = b^2 \mathbb{E}\left[g''(X)\right]$ for every function g with g, g' absolutely continuous and g, g', g'' bounded. This result is interesting in its own right as it involves a second order differential operator and we show that it can be obtained by iterating several more traditional methods for finding characterizing equations. We then introduced a

distributional transformation which has the Laplace distribution as a fixed point and can be realized in terms of the zero bias transformation, and used this to derive general bounds on the distance between the law of a random variable X and the Laplace$(0, b)$ distribution. Finally, inspired by work of Peköz and Röllin on exponential approximation [30], we applied these results to construct general bounds on the distance of various random sums to the Laplace distribution. As a special case, we proved

Theorem 8. *Let $X_1, X_2, \ldots$ be a sequence of independent random variables with $\mathbb{E}\left[X_i\right] = 0$, $\mathbb{E}\left[X_i^2\right] = 2b^2$, $\sup_i \mathbb{E}\left[|X_i|^3\right] = \rho < \infty$, and let $N \sim \text{Geometric}(p)$ be independent of the $X_i's$. Then*

$$d_{BL}\left(\mathscr{L}\left(p^{\frac{1}{2}}\sum_{i=1}^{N}X_i\right), \text{Laplace}(0, b)\right) \le p^{\frac{1}{2}}\frac{b+2}{b}\left(b\sqrt{2} + \frac{\rho}{6b^2}\right)$$

where d_{BL} denotes the bounded Lipschitz metric.

Future Directions.

I am ultimately interested in developing a deeper understanding of the convergence behavior of various stochastic processes and establishing new techniques to quantify convergence rates. My general research program is based on exploring these issues through concrete examples. In both Markov chain theory and Stein's method there are a plethora of open problems, ranging from longstanding questions that seem to require deep new insights to those which are likely amenable to classical techniques and careful analysis, making for excellent undergraduate research projects.

In the case of Markov chains, there have been numerous advances over the course of the last several decades (see [15, 23, 29]), but many problems of interest remain unsolved. For instance, though lower bounds on variation distance are often fairly straightforward in specific examples, more general techniques such as Wilson's method [40] would be extremely useful. Also, many approaches have been developed for the analysis of reversible chains, and it would be nice to have more methods of extending them in cases where one no longer has the spectral theorem at their disposal. A good starting point for these investigations is to consider chains in which the transition operator is diagonalizable but not necessarily normal, as in hyperplane walks [8] and Hopf-power chains [17]. Perhaps the most glaring open problem concerns establishing necessary and sufficient conditions for cut-off in total variation. Such conditions are known for L^p cut-off with $1 < p < \infty$ [12] and there has been some exciting progress as of late [5], but we still do not understand what lies at the heart of this remarkable phenomenon.

In my investigations of hyperplane walks, I was able to get a lot of mileage out of "lumping" or projecting chains onto smaller state spaces and have since become quite interested in the dependence relations between eigenvectors arising from lumping according to different equivalence relations. In the case of hyperplane walks, such knowledge may yield a proof of diagonalizability in a more probabilistic context. I also found an analogous method of state space reduction which allows one to recover left eigenvectors in examples such as random walks on finite groups that I would like to further explore. Finally, I was able to formulate some hyperplane walks in terms of "product chains" and slightly generalize that notion. I would like to extend this generalization and better understand the relations between products and projections [32].

There is also quite a bit more to be done with the theory of random walks on abelian sandpiles. We have established some general bounds that are of the correct order in certain examples, but we need more refined estimates (possibly involving additional graph parameters) to obtain sharp rates for a larger class of graphs. Furthermore, we would like to adapt our methods to answer structural questions about sandpile groups and to explain various phenomena that have been found in specific cases, such as the multiple time scales from [38].

There are several other interesting Markov chains that I am thinking about at present: the random walks on irreducible representations of finite groups from [18, 20], the "flip chains" on Catalan objects studied in [26, 28], and the random-to-random shuffle mentioned in the introduction. In each of these examples, part of the fun lies in the Markov chain theory itself and part in the combinatorics of the objects involved. There seems to be no end to the list of such problems, and each contains new challenges and surprises.

As with Markov chains, Stein's method provides a rich source of theoretical and applied problems, with the latter often providing insight into the former. On the applied side, techniques from Stein's method are extremely useful in a wide variety of settings. For example, my most recent project, joint with Alexander Bendikov and Anton Braverman [6], concerns an empirical process associated with the eigenvalues of the perturbed hierarchical Laplacians introduced in [7]. These objects are analogues of the famous Anderson Hamiltonians studied in connection with the phenomenon of localization, and one is interested in the number of eigenvalues falling in a given sequence of intervals. The crux of the problem lies in understanding certain sums of indicators with long-range dependence, and we were able to use a Stein's method result from [1] (along with a good deal of delicate analysis) to prove Poisson convergence with quantitative rates in total variation under quite mild assumptions. As a concrete example, we considered perturbations of the fractional derivative operator $\mathfrak{D}^\alpha$, $\alpha > 1$, over the field $\mathbb{Q}_p$ of p-adic numbers.

In addition to bounding error terms or convergence rates in specific applications, there are many distributions for which no Stein's method framework yet exists, and it is likely that working out the details for some of these particular cases will yield general insights into the method as a whole. One of the biggest present challenges is to find more unifying perspectives on the construction of Stein identities and bounds on the corresponding solutions. The article [25] is a recent example of such an undertaking and I am interested in pursuing a similar approach in the near future.

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